which depends on two variable parameters p and ϕ. By changing these arbitrarily a two parameter system of quartics is obtained. The quartics corresponding to all other positions of X and X' result from linear transformations of the first system.

It can easily be established that all curves of system (1) have the three lines a_1, a_2, a_3 with the equations

$$y=x+10-\frac{a}{c}\cos\phi, \quad y=-x+10+\frac{a}{c}\cos\phi, \quad x=-\frac{d}{c}$$

as common asymptotes. In Fig. 1, which represents an isometric projection of the regulus and a curve of the system, $a=1$, $b=-2$, $c=1$, $d=2$, $p=4$, $\phi=45°$. Only one complete string, $A'AP$, has been drawn, and the part of the curve which is mechanically possible is indicated by its ordinates. In order to obtain the complete curve as represented by the equation the part shown must be reflected on the line s.

3. In subways with circular cross sections and lined with glazed tiling one may observe peculiar curves of reflexion emenating from the incandescent lights and winding on the surface in an ultimate direction towards the eye of the observer.*

To study the nature of these curves, assume in Fig. 2, a perspective view, L as a source of light near the cylindrical surface of the subway and B as the eye of the observer.

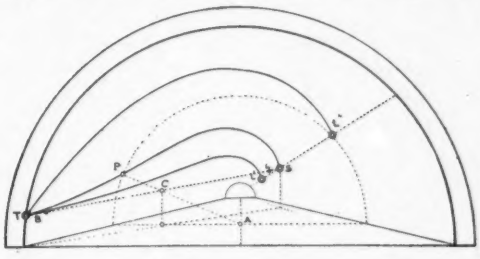

Fig. 2.

It is clear that every ray emanating from L that strikes the surface at P and is reflected to B is coplanar with the reflected ray PB. Furthermore, the perpendicular of incidence is a normal to the surface and consequently passes through the axis of the cylinder. Hence, in order to obtain points of reflexion P, pass a pencil of planes through the line connecting L and B. Each of these planes cuts the axis of the cylinder in a point A, and the surface normal from A, cutting LB at C, meets the surface in a point P of the curve containing reflecting points.

All normals PA from points of the curve pass through LB and the axis of the cylinder and are parallel to the planes normal to the axis; they form therefore an hyperbolic paraboloid. *The curve thus appears as the intersection of this regulus and the cylinder and is consequently of the fourth order.*

* Seen in the "Traforo," a subway under the Royal Gardens of the Quirinale in Rome.

It passes through the piercing points S and T, near L and B, of LB with the cylindrical surface. Every point of the cylinder gives rise to such a curve, so that a doubly infinite number of curves is obtained which all pass through T. In Fig. 2, three lamps at L, L', L'' at equal intervals and their corresponding curves are shown.

A number of problems suggest themselves in connection with a detailed study of these curves. As an example the result may be stated, that the locus of the foci of all conics (ellipses) cut out on the surface by the planes of the pencil through LB is a twisted quartic, whose orthogonal projection on a plane normal to LB passes through the circular points.

Referring again to Fig. 2, it is clear that the quartic through S and T is determined by ST and does not depend on the position of L and B on ST. Hence assuming on ST, L and B arbitrarily, there are generally only a limited number of points on the quartic where reflexion towards B takes place, so that geometrically there are no continuous curves of reflexion.

The reason why such curves are seen physically lies in the fact that in reality the tunnel-surface consists of rectangular plane pieces and the source of light of a luminous body. In place of an incident and reflected ray at each point of the quartic we therefore have bundles of rays striking the plane portions around each point of the curve. On account of the proximity of L and B to the surface, parts of each bundle will be reflected to B.

ON COMPOSITE NUMBERS P WHICH SATISFY THE FERMAT CONGRUENCE $a^{P-1} \equiv 1 \bmod P$.*

By R. D. CARMICHAEL, Indiana University.

Professor J. H. Jeans† has discussed the question of the converse of Fermat's theorem, showing that the relation

$$a^{P-1} \equiv 1 \bmod P, \tag{1}$$

which (by Fermat's theorem) is always true when P is a prime for any value of a which is prime to P, is for any particular value of a true for values of P which are not prime. Mr. E. B. Escott‡ has given a more direct proof of the same theorem.

The failure of the converse of Fermat's theorem has also been pointed out by Lucas§ by means of the example

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* Read before the American Mathematical Society, October 28, 1911.

† *Messenger of Mathematics*, 27 (1897-8), p. 174.

‡ *Messenger of Mathematics*, New Series, No. 431 (1907), p. 175.

§ *Theorie des nombres*, I, p. 422.

$$2^{37.73-1} \equiv 1 \bmod 37.73.$$

Lucas states the true converse in this form: *If a^x-1 is divisible by P for $x=P-1$, but for no other value of x which is a divisor of $P-1$, then P is a prime number.*

A detailed study of (1) for composite P has been made by Cipolla.* Among other things he shows that for every a there exists an infinite number of composite integers P satisfying (1). Conversely, for every odd P, not a power of 3, the congruence (1) is always satisfied by some number a different from ± 1.

The object of the present note is to extend the preceding results by proving the theorem that there are values of composite P for which relation (1) is true when a is any number prime to P. A necessary and sufficient condition for this will be given and a method will be explained for obtaining the appropriate values of P. The note is an amplification of an earlier remark by the present writer.†

Let a function $\lambda(m)$, where

$$m=2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n},$$

and the numbers $p_1, p_2, \ldots, p_n$ are different odd primes, be defined in the following manner:

$$\lambda(2^{\alpha})=\phi(2^{\alpha}), \text{ if } \alpha=0, 1, 2;$$
$$\lambda(2^{\alpha})=\tfrac{1}{2}\phi(2^{\alpha}), \text{ if } \alpha>2;$$
$$\lambda(p_i^{\alpha_i})=\phi(p_i^{\alpha_i}), \text{ when } p_i \text{ is an odd prime;}$$
$$\lambda(m)=\text{least common multiple of } \lambda(2^{\alpha}), \lambda(p_1^{\alpha_1}), \ldots, \lambda(p_n^{\alpha_n}).$$

Then it is well known that for every a prime to P we have the congruence

$$a^{\lambda(P)} \equiv 1 \bmod P. \tag{2}$$

Furthermore, it has been proved‡ that $\lambda(P)$ is the least exponent such that (2) is true for every a prime to P. Hence, it follows at once that if (1) is true $\lambda(P)$ must be a factor of $P-1$. Again, if $\lambda(P)$ is a factor of $P-1$ the relation (1) is satisfied for every a prime to P. Hence the following theorem:

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* *Annali di Matematica* (3) 9 (1903), pp. 139-160.

† *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 237-238.

‡ *Bulletin of the American Mathematical Society*, Vol. 16 (1910), pp. 232-238.

THEOREM I. *A necessary and sufficient condition on the integer P in order that the congruence*

$$a^{P-1} \equiv 1 \bmod P$$

shall be true for every a which is prime to P is that $P-1$ shall be divisible by $\lambda(P)$; or

$$P-1 \equiv 0 \bmod \lambda(P). \tag{3}$$

From this theorem it follows at once that P and $\lambda(P)$ are relatively prime. Hence, P does not contain a repeated prime factor; for, if so, such a prime would be a factor both of P and of $\lambda(P)$—which we have just seen to be impossible. Moreover, P cannot be a product of two prime factors; for if $P=pq$ and $p>q$, it follows from theorem I that

$$\frac{pq-1}{p-1} = \text{integer}.$$

But

$$\frac{pq-1}{p-1} = q + \frac{q-1}{p-1}.$$

Since p is greater than q the second member of the last equation is not an integer. That is, $P=pq$ does not in any case satisfy theorem I. Now, $\lambda(P)$ is even since $\lambda(m)$ is even when $m \neq 2$ and P is composite so that it is not 2. Hence (3) cannot be satisfied by an even P. Collecting these results, we have

THEOREM II. *In order that composite P shall satisfy the congruence*

$$a^{P-1} \equiv 1 \bmod P$$

for every a which is prime to P it is necessary that a shall be the product of three or more different odd prime factors.

We shall now prove the following theorem:

THEOREM III. *There are values of composite P for which the congruence*

$$a^{P-1} \equiv 1 \bmod P$$

is true when a is any integer prime to P.

We shall prove this theorem by actually finding numbers P of the form

$$P=pqr$$

which satisfy the necessary and sufficient condition of theorem I, the numbers p, q, r being primes.

Evidently, a necessary condition that $P=pqr$ shall satisfy (3) is that each of the expressions

$$\frac{pqr-1}{p-1}, \quad \frac{pqr-1}{q-1}, \quad \frac{pqr-1}{p-1}$$

shall be an integer. Subtracting from these numbers in order the integers qr, rp, pq we have the result that the remainders

$$\frac{qr-1}{p-1}, \quad \frac{pr-1}{q-1}, \quad \frac{pq-1}{r-1} \tag{4}$$

must each be an integer. We shall refer to this as condition (4).

If $p=3$ it is easy to show that there is the single number 3.11.17 which satisfies condition (4), but that this number fails to satisfy the condition in theorem I. For $p=5$ two solutions satisfying (4) and theorem I are found; namely, 5.13.17 and 5.17.29. For $p=7$ we have the four solutions

7.13.19, 7.13.31, 7.19.67, 7.31.73.

We shall illustrate the method of finding solutions by carrying out the process in detail for the case $p=7$.

For this case the first number in (4) is

$$\frac{qr-1}{6}.$$

In order that this shall be an integer it is necessary and sufficient that both q and r shall be of the form $6n+1$ or both of the form $6n-1$. From the third number in (4) we see that

$$\frac{7q-1}{r-1}=m,$$

where m has one of the values 2, 3, ..., 6, since r is greater than q. This equation gives

$$r=\frac{7q+m-1}{m}. \tag{5}$$

Substituting this value of r in the second member of (4) we find that we must have

$$\frac{4pq+6m-7}{m(q-1)}=\text{integer}=\frac{1}{m}\left(49+\frac{6m+42}{q-1}\right).$$

The values of q which satisfy this relation are the following:

For $m=2$, $q=19$;
For $m=3$, $q=13, 31$;
For $m=4$, $q=23$;
For $m=5$, $q=13, 73$;
For $m=6$, No value of q.

If we substitute in (5) and remember that r must be prime we see that $q=23$ and $q=73$ are both impossible. The other values of q in order give the numbers

$$7.19.67,\quad 7.13.31,\quad 7.31.73,\quad 7.13.19$$

as the only possible values of P which are of the form $7qr$. Testing these values by means of theorem I, we see that each of them is a possible value of P as was stated above.

In a similar way we may assume other values of p and determine all possible values of P having such a prime factor p and having the desired property of satisfying (1) for every a prime to P. The modification of the method which is necessary for dealing with P as the product of four or more different odd primes is obvious. In this way one determines the following integers P^* having the property that the congruence

* This list might be indefinitely extended.

$$a^{P-1} \equiv 1 \bmod P$$

is true for every a which is prime to P:

5.13.17	13.37.241
5.17.29	13.37.97
7.13.19	13.37.61
7.13.31	31.61.271
7.19.67	31.61.211
7.31.73	31.61.631
13.61.397	37.73.109

13.37.73.457

ON REMARKABLE POINTS OF CURVES.

By S. LEFSCHETZ, University of Nebraska.

Among the points that we define below as *remarkable*, the only ones that have been considered to any serious extent, are the well known singular points of algebraic curves. Plücker* in solving the famous Poncelet pardox on the class of the reciprocal of an algebraic curve, gave the formula connecting their numbers. Caily† subsequently gave analogous ones for twisted curves, and Veronese‡ extended his results for n-space. In what follows an attempt is made to make precise the notion of remarkable point of a curve, by defining a class of points that can be reasonably so called, and two very simple propositions concerning these points are established. It is proper to remark that the discussion is not restricted to algebraic curves.

Let us consider in a plane an innumerable aggregate A of points, and the innumerable aggregate B of lines obtained by joining any two points of the aggregate A.

* *Algebraische Kurven.*

† *Coll. Math. Papers*, Vol. 1, p. 207.

‡ *Behandlung über die Methode des Projicirens und Schneidens, Math. Ann.* Vol. 19, p. 213.

Let C be a real curve given arbitrarily in the plane, subjected to the only condition that it shall have an analytic arc. Next consider the net of curves of order m passing through λ points of A and tangent to μ lines of B. For given values of m, λ, μ, it is clear that we have only an innumerable aggregate of these nets, since each net is a function of a finite number of points or lines taken from a discrete aggregate of them. In fact each net can be made to correspond to a series of terms depending upon $(\lambda+\mu)$ indices, and these evidently form an innumerable aggregate. Taking a definite net there is in general a number k such that at any point M of C there is a curve of the net having with C a contact of order k but none having with it a contact of order $(k+1)$. For some points of C, part of an innumerable aggregate, there will be at least one curve of the net having with C a contact of order $(k+1)$ at these points, and also some curves of the net will have with C a contact of order k in more than one point. It would be conceivable that all curves of the net having a contact of order k with C, satisfy this condition for h points, in which case we would single out those having a contact of order k in $(h+1)$ points. The points where there is a contact of order $(k+1)$ with a curve of the net, and the points where C is touched by the curves that have $(h+1)$ times a contact of order k with it will be called *remarkable points*, and we will include in this definition, the points of C corresponding to the remarkable points of its reciprocal with respect to an arbitrary conic of its plane. For given values of (m, λ, μ) it is clear that the points in question form a discrete aggregate on C, and therefore the same is true when m, λ, μ take all integral values. Hence, *the remarkable points of a curve form a discrete aggregate on it.*

Let O be an arbitrary point in the plane of C, and join to it all remarkable points of C, thus obtaining an innumerable aggregate of rays. As the lines going through O form an aggregate having the power of the continuum, if we subtract from it the above discrete aggregate of rays, the remaining one has still the power of the continuum.* Hence, *the aggregate of lines cutting a curve in none of its remarkable points has the power of the continuum.* This proposition would clearly be unmodified if instead of straight lines we took any algebraic curve.

The remarkable points as above defined include evidently a great variety of points seldom associated. Thus vertices of a conic are points of hyperosculation for the circles of curvature of the curve. There the aggregate A is formed of the two circular points at infinity. A multiple point is the reciprocal of points of contact of a multiple tangent of the reciprocal polar of C, and therefore is also included. In this case $A=B=0$; a multiple point is a remarkable point independent of the aggregate A.

Finally, we may say that the extension to n-space varieties is quite evident.

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* E. Borel, *Lecons sur la Theorie des Fonctions*, p. 18.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

360. Proposed by CHARLES C. GROVE, Columbia University, New York.

A bridge club of 28 members has 27 meetings. There are 7 tables with 4 members at each table. Can the players be so arranged that at the end of the season (27 meetings) each member will have played *with* every other member one game and *against* every other member two games, one game meaning one meeting; and how?

Solution by the PROPOSER.

The number of combinations of 28 people two at a time, ${}^{28}C_2$, is 14 times 27; of which each individual appears in ${}^{27}C_1$ or 27.

Thus each member plays *once with* each other member. At each of the 27 meetings 14 combinations of two each are playing, and so in the season all the possible combinations will be used, 14×27. Further, at each meeting any member is playing *with* one member and *against* two other members. Since he thus goes around the membership *once with* each he will be going around *twice against each.*

The method of arranging the players can be seen from the following:

1, 2	2, 3	3, 4	4, 5	...	26, 27	27, 28
1, 3	2, 4	3, 5	4, 6	...	26, 28	
1, 4	2, 5	3, 6	⋮			
1, 5	⋮	⋮	4, 28			
1, ⋮	⋮	3, 28				
⋮	2, 28					
1, 28						

where the numbers represent the different members of the club.

At the various meetings the tables must be filled from the above combinations.

Meeting	Table I.	II.	III.	IV.	V.	VI.	VII.
	1	5	9				
First	3 4	7 8	11 12	. . . .			
	2	6	10				
	1	5	9				
Second	2 4	6 8	10 12	. . . .			
	3	7	11				

	1	5	9	
Third	2 3	6 7	10 11	. . . .
	4	8	12	

All the combinations involving 1, 2, 3, 4 only have been used, and thereby 1 has played *with* each 2, 3, 4 once and twice against each. Since the players are identically involved, *mutatis mutandis*, all the other players will have a similar experience.

361. Proposed by C. E. GITHENS, Ph. D., Wheeling, W. Va.

Find three integral values for $[-10+9\sqrt{(-3)}]^{\frac{1}{3}}+[-10-9\sqrt{(-3)}]^{\frac{1}{3}}$. A solution not involving a cubic is desired.

I. Solution by the PROPOSER.

I. Put $(-10+9\sqrt{-3})^{\frac{1}{3}}+(-10-9\sqrt{-3})^{\frac{1}{3}}=\frac{1}{2}[(-80+72\sqrt{-3})^{\frac{1}{3}}+(-80-72\sqrt{-3})^{\frac{1}{3}}]$.

Let $(-80+72\sqrt{-3})^{\frac{1}{3}}=\sqrt{x}+\sqrt{y}$ and $(-80-72\sqrt{-3})^{\frac{1}{3}}=\sqrt{x}-\sqrt{y}$.

Then, $[6400-(-15552)]^{\frac{1}{3}}=x-y=28$, and, therefore, $(-80+72\sqrt{-3})^{\frac{1}{3}}=\sqrt{(x+28)}+\sqrt{x}$. Hence, by raising both sides to the third power, $(4x+28)\sqrt{(x+28)}+(4x+84)\sqrt{x}=-80+72\sqrt{-3}$.

Put $4x+28\sqrt{(x+28)}=-80$, and $4x+84\sqrt{x}=72\sqrt{(-3)}$.

Let $4x+28=-80$ or factor of -80; -40, -20, -16, -10, -9, -8, -5, -4, -2, and

$4x+84=72\sqrt{(-3)}$ or factor of 72; 36, 24, 18, 12, 9, 8, 6, 4, 3, 2.

Subtracting, $-56=-20-(36)$.

$\therefore 4x+28=-20$ and $x=-12$.

[1] $\frac{1}{2}[(-80+72\sqrt{-3})^{\frac{1}{3}}+(-80-72\sqrt{-3})^{\frac{1}{3}}]$

$=\frac{1}{2}[(\sqrt{(x+28)}+\sqrt{x})+(\sqrt{(x+28)}-\sqrt{x})]=4$, answer.

II. Similarly with $(-80+72\sqrt{-3})^{\frac{1}{3}}=\sqrt{(x_0+28)}-\sqrt{x_0}$, in which the factors -80 and 24 added to eliminate the $4x$'s produces

$x_0=-27$ and $\frac{1}{2}[(\sqrt{(x_0+28)}-\sqrt{x_0})+\sqrt{(x_0+28)}+\sqrt{x_0}]=-1$, answer.

III. With $-\sqrt{(x_1+28)}+\sqrt{x_1}$ and -16 and 72 as factors as above, $x=-3$ and $\frac{1}{2}[-\sqrt{(x_1+28)}+\sqrt{x_1}+(-\sqrt{(x_1+28)}-\sqrt{x_1})]^{\frac{1}{3}}=-5$, answer.

IV. The root $-\sqrt{(x_2+28)}-x_2$ produces no two factors of -80 and 72 whose difference or sum equals ± 56; hence it is not a root, which is as it should be, for the numerical equation is an example of the "irreducible case" in the Cardan solution of a cubic whose equation is $x^3-21x+20=0$.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Maryland.

Putting $(-10+9\sqrt{3}.\sqrt{-1})=\rho(\cos\phi+\sin\phi.\sqrt{-1})$, we get $\tan\phi=-\frac{9}{10}\sqrt{3}$. $\rho=\sqrt{343}$, and $(-10+9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}+(-10-9\sqrt{3}.\sqrt{-1})^{\frac{1}{3}}=2\sqrt{7}\cos\frac{1}{3}\phi$.

$\therefore 4\cos^3\frac{1}{3}\phi-3\cos\phi+\frac{10}{49}\sqrt{7}=0$.

By trial, $\cos\frac{1}{3}\phi=\frac{2}{7}\sqrt{7}$, and dividing the last trinomial by $\cos\frac{1}{3}\phi-\frac{2}{7}\sqrt{7}$, we get $4\cos^2\frac{1}{3}\phi+\frac{8}{7}\sqrt{7}.\cos\frac{1}{3}\phi-\frac{5}{7}=0$; whence $\cos\frac{1}{3}\phi=-\frac{5}{14}\sqrt{7}$, $\cos\frac{1}{3}\phi=\frac{1}{14}\sqrt{7}$.

$\therefore$ The three values required are 4, -5, 1.

Also solved by A. M. Harding.

GEOMETRY.

386. Proposed by DANIEL KRETH, Oxford, Iowa.

Construct the triangle, having given, the vertical angle, the sum of the three sides, and the perpendicular.

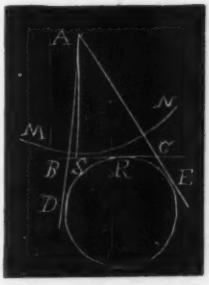
I. Solution by H. PRIME, Boston, Massachusetts.

Let ABC be the required triangle, C the given angle. On AB produced take $BE=BC$. On BA produced take $AF=AC$. Let O be the center of circle ECF. Then we have the angles $FOE=2(BEC+AFC)=ABC+BAC$=supplement of C.

Hence, to construct the triangle, on EF=the given sum of the three sides form the isosceles triangle EOF, making EOF=the supplement of the given vertex angle (or $OEF=OFE$=one half the given angle). About O as center describe the arc EF. Parallel to EF and at a distance from it equal to the given altitude draw a line meeting the arc at C and C'. Draw CA and CB, making the angles $ACF=AEC$ and $BCE=BEC$. ABC is the required triangle.

II. Solution by C. N. SCHMALL, New York City, and A. M. HARDING, University of Arkansas.

Construct an angle A equal to the given vertical angle. Lay off AD and AE each equal to *half* the given sum of the sides. Describe a circle touching these lines in D and E. With A as center and radius equal to the given perpendicular, describe a circle MN. By a well known method draw a line tangent to *both* these circles touching in R and S, respectively, and cutting the sides in B and C. Then ABC is the triangle required.



Proof. $BR=BD$, $CR=CE$.

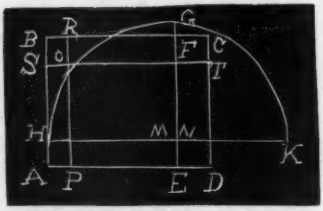
$\therefore BC=BD+CE$; hence the triangle has the given perimeter. Also, AS is perpendicular to BC; therefore the triangle has the required altitude. Q. E. D.

Also solved by J. Scheffer and A. H. Holmes.

387. Proposed by DANIEL KRETH, Oxford, Iowa.

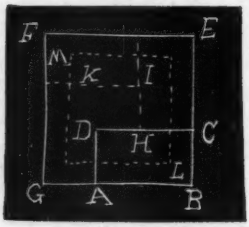
A lot 100 feet long and 60 feet wide, has a walk extending from one corner half way around it, and occupying one-third of the area. Required the width of the walk. A geometrical construction is desired.

I. Solution by A. H. HOLMES, Brunswick, Maine.



Let $ABCD$ be the lot. $AB=CD=60$, and $AD=BC=100$. On DA take DE=one third of CD, and draw EF parallel to CD and cutting BC in F. On EF take EN=one fourth of DE and draw HN parallel to AD, the point H being on AB and $AH=EN$. On HN extended take $NK=NF$. Bisect HK in M, and with MH as radius describe a semi-circle on HK. Extend EF to cut the circumference in G. On EA take $EP=NG$. Draw PR parallel to AB. Take $BS=AP$, and draw ST parallel to BC. Then on ST take $SO=AP$. Then $ABRP+ORCT=\frac{1}{3}ABCD$, and AP =width of walk required; which is shown as follows: Let x=width of walk. Then $100x+(60-x)x=2000$. Therefore, $(80-x)^2=55\times 80$.

II. Solution by C. E. GITHENS, Ph. D., Wheeling, West Virginia.



Let $ABCD$ be the given lot. Form the square $GBEF$ by arranging three other equal lots as in the figure. Then $GB=60$ feet$+100$ feet $=160$ feet.

Area of square $DHIK=40^2$ square feet

Area of square $LM=\frac{3}{3}(4ABCD)+DHIK=$ 17,600 square feet.

Hence, side of square $LM=\sqrt{(17,600)}$ feet $=40\sqrt{11}$ feet.

Hence, width of walk$=\frac{1}{2}(160-40\sqrt{11})$ feet$=(80-20\sqrt{11})$ feet.

Also solved by H. Prime and S. G. Barton.

388. Proposed by WILLIAM HOOVER, Ph. D., Professor of Mathematics and Astronomy, Ohio University Athens, Ohio.

A conic is inscribed in a triangle and one focus lies on the polar circle of the triangle. Prove that the corresponding directrix passes through the center of perpendiculars.

Solution by the PROPOSER.

Reciprocating with respect to the focus, the conic corresponds to the circumscribing circle of the reciprocal triangle; the polar circle, whose center is the orthocenter of the fixed triangle, to a parabola with focus, the fixed focus of the given conic, the given orthocenter to the directrix of the

reciprocal parabola, the directrix of the conic to the center of the reciprocal circle which is on the directrix of the parabola.

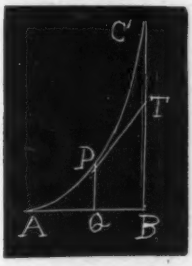
CALCULUS.

314. Proposed by REV. J. H. MEYER, S. J., New Orleans, La.

A fox started from a certain point and ran due east 300 yards, when it was overtaken by a hound that started from a point 100 yards due north of the fox's starting point, and ran directly towards the fox throughout the race. Find the lengh of the curve described by hound, both having started at the same instant, with a uniform velocity.

Solution by J. SCHEFFER, A. M., Hagerstown, Maryland, and FRANCIS E. RUST, E. E., Pittsburg, Pa.

Let A be the starting point of the hound, and B that of the fox, C the point of capture, P some point of the curve described by the hound, $AQ=x$, $PQ=y$, DPT a tangent to the curve at P, $AB=a$ $(=100)$, $BC=b(=300)$; $AP=s$, m=rate of hound, n= rate of fox; $s=mt$, $BT=nt$, t being a certain time.



$BT=y+(a-x)\tan TBE=y+(a-x)\dfrac{dy}{dx}$.

$$\therefore \frac{s}{m}=\frac{y+(a-x)(dy/dx)}{n}\text{; or, putting }\frac{n}{m}=\beta,\ \beta s=$$

$y+(a-x)\dfrac{dy}{dx}$. Differentiating, $\beta\dfrac{ds}{dx}=(a-x)\dfrac{d^2y}{dx^2}$; or

$$\beta\sqrt{1+\left(\frac{dy}{dx}\right)^2}=(a-x)\frac{d^2y}{dx^2}.\quad\text{Putting }\frac{dy}{dx}=p,\ \beta\sqrt{(1+p^2)}=(a-x)\frac{dp}{dx}.$$

$$\therefore \beta\frac{dx}{a-x}=\frac{dp}{\sqrt{(1+p^2)}}.\quad\therefore\frac{C_1}{(a-x)^\beta}=p+\sqrt{(1+p^2)},\text{ and since for } x=0,$$

$p=0$, $\dfrac{C_1}{a^\beta}=1$, and $C_1=a^\beta$. Now, $\dfrac{a^\beta}{(a-x)^\beta}=p+\sqrt{(1+p^2)}$, or $a^\beta(a-x)^{-\beta}=$

$p+\sqrt{(1+p^2)}$; whence $p=\dfrac{dy}{dx}=\frac{1}{2}[a^\beta(a-x)^{-\beta}-a^{-\beta}(a-x)^\beta]$.

$$\therefore y=\frac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}-\frac{a^\beta(a-x)^{1-\beta}}{1-\beta}+C_2\text{; but } 0=\tfrac{1}{2}\left(\frac{a^{-\beta}a^{1+\beta}}{1+\beta}-\frac{a^\beta a^{1-\beta}}{1-\beta}\right)+$$

C_2, whence $C_2=\dfrac{a^\beta}{1-\beta^2}$.

$$\therefore y=\tfrac{1}{2}\left[\frac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}-\frac{a^\beta(a-x)^{1-\beta}}{1-\beta}\right]+\frac{a^\beta}{1-\beta^2};\quad\frac{ds}{dx}=\tfrac{1}{2}[a^\beta(a-x)^{-\beta}+$$

$a^{-\beta}(a-x)^\beta]$;

$$\therefore s=-\tfrac{1}{2}\left[\frac{a^{\beta}(a-x)^{1-\beta}}{1-\beta}+\frac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}\right]+C_3. \quad \text{For } x=0,\ s=0.$$

$$\therefore C_3=\frac{a}{1-\beta^2}. \quad \therefore s=\frac{a}{1-\beta^2}-\tfrac{1}{2}\left[\frac{a^{\beta}(a-x)^{1-\beta}}{1-\beta}+\frac{a^{-\beta}(a-x)^{1+\beta}}{1+\beta}\right].$$

Since for $x=a$, $y=b$, we have $b=\frac{a^{\beta}}{1-\beta^2}$; or, substituting $a=100$, $b=300$, we have $\beta^2+\frac{1}{3}\beta=1$; whence $\beta=\frac{1}{6}(\sqrt{37}-1)$ and length of curve between A and $C=\frac{a}{1-\beta^2}=50(\sqrt{37}+1)=354.135$.

$$y=\frac{na}{n^2-1}-\frac{na^{1/n}}{2(n-1)}(a-x)^{(n-1)/n}+\frac{n}{2(n+1)}\frac{1}{a^n}(a-x)^{(n-1)/n}.$$

315. Proposed by C. N. SCHMALL, New York City.

If $y=f(x)$, show by Taylor's Theorem that

$$f\left(\frac{x}{1+x}\right)=y-\frac{x^2}{1+x}\cdot\frac{dy}{dx}+\frac{x^4}{2(1+x)^2}\cdot\frac{d^2y}{dx^2}-\frac{x^6}{2.3.(1+x)^3}\cdot\frac{d^3y}{dx^3}+\text{... etc.}$$

Solution by the PROPOSER.

Put $\frac{x}{1+x}=x+h$, then $f\left(\frac{x}{1+x}\right)=f(x+h)$,

and also, $h=\frac{x}{1+x}-x=-\frac{x^2}{1+x}$.

$\therefore h^2=\frac{x^4}{(1+x)^2}$, $h^3=-\frac{x^6}{(1+x)^3}$, and so on.

Substituting these values of the powers of h in Taylor's series, we have the required result; *i. e.*, from

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\text{etc.},$$

we get, by direct substitution,

$$f\left(\frac{x}{1+x}\right)=y-\frac{x^2}{1+x}\frac{dy}{dx}+\frac{x^4}{2(1+x)^2}\frac{d^2y}{dx^2}-\frac{x^6}{2.3(1+x)^3}\frac{d^3y}{dx^3}+\text{etc.}$$

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^{\infty}\frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^{\infty}\frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

Solution by A. M. HARDING, Fayetteville, Arkansas.

It can easily be shown that

$$\int_0^\infty e^{-bz}\cos bm\,db = \frac{x}{m^2+x^2};\quad \int_0^\infty e^{-bz}\sin bm\,db = \frac{m}{x^2+m^2}$$

$$\text{and } 2\int_0^\infty be^{-b^2(1+x^2)}db = \frac{1}{1+x^2}.$$

Hence $u = \int_0^\infty \frac{x\sin ax}{1+x^2}dx = \int_0^\infty \sin ax\left(\int_0^\infty e^{-bz}\cos bdb\right)dx.$

$$\therefore \frac{du}{db} = \int_0^\infty \sin axe^{-bz}\cos bdx = \cos b\int_0^\infty e^{-bz}\sin axdx = \cos b.\frac{a}{a^2+b^2}.$$

$$\therefore u = \int_0^\infty \frac{a\cos b}{a^2+b^2}db, \text{ putting } ax \text{ for } b, = \int_0^\infty \frac{\cos ax}{1+x^2}dx.$$

$$\therefore \int_0^\infty \frac{x\sin ax}{1+x^2}dx = \int_0^\infty \frac{\cos ax}{1+x^2}dx.$$

Now $u = \int_0^\infty \frac{\cos ax}{1+x^2}dx = \int_0^\infty \cos ax\,(2\int_0^\infty be^{-b^2(1+x^2)}db)\,dx.$

$$\frac{du}{db} = 2\int_0^\infty \cos ax.be^{-b^2(1+x^2)}dx = 2\,be^{-b^2}\int_0^\infty e^{-b^2x^2}\cos ax\,dx = 2be^{-b^2}.\frac{\sqrt{\pi}}{2b}e^{-(a^2/4b^2)}$$

(see Byerly's *Integral Calculus*, Art. 93 (b))

$$= \sqrt{(\pi)}e^{-b^2-[(a/2)^2/b^2]}.\quad u = \sqrt{(\pi)}\int_0^\infty e^{-b^2-[(a/2)^2/b^2]}db = \sqrt{(\pi)}.e^{-a}.\frac{\sqrt{\pi}}{2} = \frac{\pi}{2}e^{-a}.$$

(Art. 93 (a)).

Also solved by J. Scheffer, S. A. Corey, and the Proposer.

NOTES AND NEWS.

Professor Cajori's *Theory of Equations*, which was published in 1904, was reprinted by the Macmillan Company during last January. M.

The spring meeting of the Chicago section of the American Mathematical Society will be held at Chicago on Friday and Saturday, April 5 and 6, 1912. S.

The third number of the *Tohoku Mathematical Journal*, published at Sendai, Japan, appeared during last January. It contains seven articles, of which six are in English while the remaining one is in German. A volume composed of four numbers, is sold for one dollar. M.

The Department of Superintendence of the National Education Association held its annual meeting in St. Louis, on February 27, 28, and 29, 1912. Many interesting questions are found on its program and on those of the various educational societies which met in St. Louis during the same week. S.

The remaining two volumes of the second edition of Pascal's *Repertorium der hoeheren Mathematik* are now expected to appear toward the end of the present year. It may be remembered that three successive circulars issued by B. G. Tuebner, of Leipzig, announced that these volumes would appear in the spring and summer of 1911 and at Easter of 1912. M.

During last September the University of Illinois published a list of serials in the university library. This includes newspapers, magazines, periodicals, and serial publications of societies, corporations, institutions, and governmental bodies. These serials have appeared under more than 8000 different titles and 215 of them are classified under mathematics, astronomy, or physics. M.

In an interesting article on non-euclidean geometry in the current number of the *Bulletin of the American Mathematical Society*, it is found by actual count that of all who have ever written on non-euclidean geometry, Dr. Halsted has the most titles to his credit, seventy-seven with his translations, which with his biography of Sommerville bring the count up to ninety-two. F.

The North Central Association of Schools and Colleges will hold its annual meeting at the Auditorium Hotel in Chicago on Friday and Saturday, March 22 and 23, 1912. An important report on the accrediting of schools will be presented by a committee of which President Hill, of the University of Missouri, and Director Judd, of the University of Chicago School of Education, are members. S.

The annual conference of the University of Chicago with its coöperating schools, which for twenty years has been held in November, will this year be held in April. The date was changed in order to give the representatives of secondary schools time to visit the junior college work of the university which they have been doing all the autumn and winter. The programme of the conference will be based upon the reports of their visitations from the schools to the university, thus reversing the usual order of procedure. S.

At the University of Illinois the following mathematical courses, beyond a first course in calculus, are now being given: Functions of a Complex variable, Professor Townsend; Theory of Numbers and Theory of Equations, Professor Miller; Theory of Statistics, Professor Rietz; Solid Analytic Geometry, Professor Sisam; Theory of Potential, Professor Shaw; Projective Geometry and Linear Transformations, Professor Emch; Functions of a Real Variable, Dr. Crathorne; Partial Differential Equations, Dr. Wahlin; Advanced Calculus, Dr. Lytle. M.

The Edinburgh Mathematical Society is publishing a brief review of elementary mathematics and science, entitled *Mathematical Notes.* The series began in April, 1909, and the nine numbers which have appeared cover 108 pages in all. No. 9 was issued in January, 1912, and contains four short papers entitled respectively: "Logarithms and the Reciprocals of Numbers; Certain Processes in the Theory of Equations Illustrated Geometrically; The Arithmetic Mean of a Number of Real Positive Numbers is not Less than Their Geometric Mean; Additional Notes on the Right-Angled Triangle. All contributions and communications referring to these notes should be addressed to P. Pinkerton, George Watson's College, Edinburgh, Scotland." M.

The sixth number of the journal published by the new Spanish Mathematical Society under the title *Revista de la Sociedad Matematica Espanola* appeared during last February. In addition to the usual articles this number contains a list of the 423 members of this young and active society. From this list it appears that the Spanish speaking countries of America are, as yet, too poorly represented in this society. As the articles appearing in the "Revista" are quite elementary and represent a number of different interests, it is to be hoped that this journal will serve to awaken a more general mathematical interest in the countries which employ the Spanish language, and that it may soon find more hearty support from this side of the Atlantic. M.

The provisional report of the national committee of fifteen on Geometry Syllabus, which was presented at the San Francisco meeting of the National Education Association, in July, 1911, and also of the meeting of the American Federation of the Teachers of the Mathematical and Physical Sciences, held in Washington in December, 1911, has since been revised by the committee in the light of many helpful suggestions received from many sources, and a large edition has been issued for general distribution. Copies will soon be mailed to those who are members of the associations belonging to the Federation. Others who desire copies may secure them *gratis* upon application to the Commissioner of Education, Department of the Interior, Washington, D. C., to whom a large number has been donated for general distribution. S.

In connection with the large French Mathematical Encyclopedia there is being published a "Tribune Publique" devoted to corrections and additions relating to the parts of this great work which have been issued. The entire mathematical world is invited to help in this way toward making this work more complete and more reliable. In 1904 G. Enestroem called attention, in the Bibliotheca Mathematica, page 398, to the usefulness of such general coöperation and to the fact that it is important to omit reference to useless or unreliable articles, as well as to give reference to all articles, which advance the subject in hand. It is to be hoped that the "Tribune publique" will receive more and more general support and that an increasing number of mathematicians will realize the dignity and importance of assisting, even in this humble manner, in this useful but colossal enterprise.
M.

From the 9th to the 15th of October, 1911, vacation courses were offered at Zurich, Switzerland, for the benefit of all the secondary teachers of Switzerland. There were 520 in attendance and forty-eight different courses were given. The following is a list of the mathematical courses, according to the January, 1912, number of L'Ensignement Mathématique: Introduction to the theory of groups, 6 hours; Astronomical observations and determinations of a position, 3 hours; The foundations of geometry, 5 hours; Vectorial analysis, 4 hours. The following three subjects were discussed: The notion of function in secondary instruction; Agreement between technical drawing and descriptive geometry; The use of certain problems of physics as applications in the teaching of mathematics. Some of these courses and subjects for discussion appear to be of a higher grade than those usually demanded by our teachers of secondary mathematics. M.

B. G. Teubner of Liepzig, Germany, has begun the publication of a series of small volumes at 20 cents each, which are intended to give an elementary exposition of various parts of elementary mathematics, and its contact with more advanced subjects. According to a recent announcement (January, 1912), the following four volumes have appeared: Numerals and number systems of ancient and modern civilized countries; The concept of number with its logical and historical significance; The Pythagorean theorem and its bearing on Fermat's theorem; and Calculus of probability with applications. The general series bears the name "Mathematische Bibliothek" and is edited by Dr. W. Leitzmann and Dr. A. Witting. It aims to enable those who are interested in mathematics in the widest sense to pursue the subject beyond what is ordinarily presented in the schools. The fact that these volumes can be sold at such a small price seems to indicate that such literature finds a large number of buyers in Germany and directs attention to the extravagant price charged in this country even for elementary text-books.
M.

BOOKS.

Tables of Physical and Chemical Constants and Some Mathematical Constants. By G. W. C. Kaye, B. A. (Cantab.), D. Sc. (Lond.), A. R. C. Sc. (Lond.), The National Physical Laboratory; late Sub-Lector in Physics, Trinity College, Cambridge, and T. H. Laby, B. A. (Cantab.), Professor of Physics, Wellington, N. Z.; formerly Exhibition of 1851 Scholar; Joule Student; and Research Exhibitioner, Emanuel College, Cambridge. Large 8vo. Flexible cloth back, vi+153 pages. Price, $1.50. New York: Longmans, Green & Co.

A desideratum of every present-day physics teacher has been the satisfying of the need of an accurate, authorative, and inexpensive set of physical and chemical tables for the laboratory. The book before us satisfies this need admirably. In this book, a half dozen pages are devoted to the various units,—absolute and derived. Then follows tables dealing, among other things, with terrestrial and astronomical constants, barometry, hydrometry, and densities, screws and wire gauges, elasticities and tensil strengths, viscosities, molecular constants and kinetic theory, critical data, diffusion, surface tensions, hygromety, and vapor pressure.

In the pages on heat, considerable attention is given to present-day thermometry, much data concerning the melting and boiling point, standard temperatures, expansive coefficients, thermal conduction, the recent determinations of Joule's mechanical equivalent, specific and latent heats, black-body radiation, the solar constants, etc. The more recently determinant value for latent heat of steam is given, viz., 540.

In the same way, we find here the various constants in sound, light, electricity, magnetism, and radio-activity. About fifteen pages are devoted to the physical constants of chemical compounds. The book also contains 4-figure logs and anti-logs, 5-figure logs e^{-x} and the more usual trigonometric functions with reciprocals and squares.

No physical nor chemical laboratory should be without these tables. They are absolutely indespensible to the wide-awake physics and chemistry teacher. F.

Plane Geometry. By C. A. Hart, Instructor in Mathematics, Wadleigh High School, New York City, and Daniel D. Feldman, head of the Department of Mathematics, Erasmas Hall High School, Brooklyn, N. Y., with the Editorial Coöperation of J. H. Tanner and Vergil Snyder, Professor of Mathematics in Cornell University. 8vo. Cloth, vii+303 pages. New York: American Book Co.

In addition to the usual theorems, demonstrations, and a large collection of exercises, human interest is added by a number of pictures of leading Geometricians, accompanied by brief biographical sketches. It will prove to be a very useful book in the hands of an enthusiastic teacher. F.

Practical Algebra, Second Course. By Joseph V. Collins, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wisconsin. 8vo. Cloth, viii+303 pages. New York: American Book Co.

In the preparation of this second course, the author had the benefit of suggestions and criticisms from a number of teachers whose practical experience in the school room enable them to give valuable aid in the production of a second book in algebra.

Many of the features of the author's First Year Course have been retained in the Second Course. The book is well written and the presentation of the principles of algebra is clearly and forcefully put before the student's mind. F.

Elementary Plane Geometry. By John C. Stone, A. M., Head of the Department of Mathematics, State Normal School, Montclair, N. J., Co-Author of the Southworth-Stone Arithmetics, Stone-Millis Secondary Arithmetics, Algebras, etc., and James F. Millis, A. M., Head of the Department of Mathematics, Francis Parker School, Chicago, Co-Author of the Stone-Millis Secondary Arithmetics and Algebras. 8vo. Red cloth sides, viii+252 pages.

In this text, much is made of every-day practical problems as examples and exercises coming under application of theorems. Also the trigonometric functions are defined and used to a very limited extent.

On page 226, Dürer's construction of a pentagon appears as exercise 20. A figure is drawn and the student is asked to explain the construction and prove that this figure is a *regular* pentagon. We wonder how the authors expect this to be done. F.

Engineering Applications of Higher Mathematics. By V. Karapetoff, Part I. Problems on Machine Design. First Edition, first thousand. 8vo. xiv+69 pages. Price, 75 cents. New York: John Wiley & Sons.

The aim of this book is not intended to present the principles of the calculus to the student, but, rather, the author tells us, to enable an engineer to make better use of his higher mathematics in his work. To this end, he has taken up a brief treatment of loads on the inclined plane, friction in journals, friction in step bearings, carrying capacity in belts, torsion of shafts, and moments of inertia of flywheels. Under each head a number of problems are solved in detail. The book will be found helpful to the practical engineer who has limited time for more extensive study. F.

Elements of Applied Mathematics. By Herbert E. Cobb, Professor of Mathematics in Lewis Institute, Chicago. 12mo. Cloth, 274 pages. Price, $1.00. Chicago: Ginn & Co.

"The idea that mathematics is a series of discrete subjects, each in turn to be studied and dropped without reference to the others, or to the mathematical problems to be met in the future, is fast being displaced by that which links arithmetic, algebra, geometry, and trigonometry in close relationship with each other, and connects all our mathematics with our work in the shops and laboratories.

Elements of Applied Mathematics is constructed upon this latter principle. The work outlined consists largely of lists of problems based on the student's preceding work in mathematics, illustrating the work in the shops and laboratories, and of simple experiments and exercises in the mathematics classroom, where the pupil, by measuring and weighing, secures his own data for numerical computations and geometrical constructions."

First Year Algebra. By William J. Milne, Ph. D., LL. D., President, New York State Normal College, Albany, N. Y. 8vo. Cloth, 320 pages. New York: The American Book Co.

This is a very excellent book for the beginner, stimulating and instructive. F.

Brief Course in Analytical Geometry. By J. H. Tanner, Professor of Mathematics, Cornell University, and Joseph Allen, Assistant Professor of Mathematics in the College of the City of New York. 12mo. Cloth and leather back, x+282+xxiv pages. New York and Chicago: The American Book Co.

This little book preserves the main features of the authors' Elementary Analytical Geometry, a book well adapted to elementary instruction in colleges. Those who have used Elementary Analytical Geometry and found it a little too comprehensive for the time at their disposal will want to try this briefer course by the same authors. F.

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ON THE SUM OF THE NUMBERS WHICH BELONG TO A FIXED EXPONENT AS REGARDS A GIVEN MODULUS.

By G. A. MILLER, University of Illinois.

§ 1. INTRODUCTION.

In article 81 of Gauss's *Disquisitiones Arithmeticae*, 1801, it is proved that the sum of the incongruent primitive roots of a prime number p is $\equiv 0 \pmod{p}$ whenever $p-1$ is divisible by the square of a prime; when $p-1$ is not divisible by the square of a prime, this sum is $\equiv 1$ or $\equiv -1$ according as $p-1$ is the product of an even or of an odd number of distinct primes. As this theorem can be so readily verified it is within easy reach of those whose mathematical attainments are very meagre. We proceed to illustrate it by using successively for p the numbers 13, 11, and 31.

The four primitive roots of 13 are 2, 6, 7, 11. Their sum is $26 \equiv 0 \pmod{13}$ and 12 is divisible by 2^2. The four primitive roots of 11 are 2, 6, 7, 8. Their sum is $23 \equiv 1 \pmod{11}$ and 10 is the product of two distinct primes. The eight primitive roots of 31 are 3, 11, 12, 13, 17, 21, 22, 24. Their sum is $123 \equiv -1 \pmod{31}$ and 30 is the product of an odd number of distinct primes.

The given theorem, due to Gauss, has been proved in various ways and was extendod by Arndt in 1846.* In Bachmann's *Niedere Zohlentheorie*, 1902, page 333, we find the following much more general theorem: The sum of the incongruent numbers which belong, with respect to mod p^α or mod $2p^\alpha$ (p being any odd prime), to any exponent which is divisible by the square of a prime is always $\equiv 0$; when this exponent is not divisible by the square of a prime, the given sum is $\equiv 1$ or $\equiv -1$ according as the exponent is the product of an even or of an odd number of distinct primes.

As It is so very easy to verify that this general theorem is not universally true it is singular that it should have appeared in such an excellent work, even if it is corrected in the Zusatze at the end of the volume. For instance, if we let $p \equiv 3$ and $\alpha \equiv 3$, it is evident that 10 and 19 are the two incongruent numbers which belong to exponent 3. Their sum is $29 \equiv 2 \pmod{27}$

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